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A note on linearization of actions of finitely semisimple Hopf algebras on local algebras

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Abstract

Let H be a Hopf algebra over a field k and let $H \otimes A \to A$, $h \otimes a \to h.a$, be an action of H on a commutative local Noetherian k-algebra (A, m). We say that this action is linearizable if there exists a minimal system $x_1, \dots, x_n$ of generators of the maximal ideal m such that $h.x_i \in kx_1 + \cdots + kx_n$ for all $h \in H$ and $i = 1, \dots, n$. In the paper we prove that the actions from a certain class are linearizable (see Theorem 4), and we indicate some consequences of this fact.

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Let k be a field and let H be a Hopf algebra over k with comultiplication $\Delta : H \to H \otimes H$ ($\otimes = \otimes_k$), antipode $S : H \to H$, and counity $\varepsilon : H \to k$. Recall that a (left) action of H on a k-algebra A is a left H-module structure $\gamma : H \otimes A \to A$ on A as a vector space over k (as usual, we write $\gamma(h \otimes a) = h.a$) such that $h.1_A = \varepsilon(h)1_A$ and $h.(xy) = \sum (h_{(1)}.x)(h_{(2)}.y)$ for all $h \in H$, $x, y \in A$, and $\sum h_{(1)} \otimes h_{(2)} = \Delta(h)$. In other words, A together with γ is an H-module algebra, see [7, 10]. Recall also that such an action is said to be locally finite if A, as an H-module, is a union of its finite dimensional submodules. If H is a finite-dimensional vector space, then clearly every action of H on a k-algebra A is locally finite.

In this paper we consider only actions of H on commutative k-algebras.

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Given an action of H on an algebra A, we say that an ideal I in A is H-invariant if $h.x \in I$ for all $h \in H$ and $x \in I$, i.e., if I is a submodule of A, as an H-module. One readily checks that if an ideal I in A is H-invariant, then all its powers I^j are also H-invariant, and so the quotient H-modules I^i/I^j, $j \geq i$, are defined.

Definition 1. An action of H on a local noetherian algebra (A,m) is called linearizable if there exists a minimal system $x_1,\ldots,x_n$ of generators of the maximal ideal m such that $h.x_i \in kx_1 + \cdots + kx_n$ for $i=1,\ldots,n$ and $h \in H$.

Remark 2. If an action of H on a local noetherian algebra (A,m) is linearizable, then it is easy to see that the maximal ideal m is H-invariant, and that for each basis $z_1,\ldots,z_n$ of m/m^2 over the quotient field A/m, there are $y_1,\ldots,y_n$ in m such that $y_i + m^2 = z_i$ and $h.y_i \in ky_1 + \cdots + ky_n$ for $i=1,\ldots,n$ and $h \in H$.

Definition 3. The Hopf algebra H is called (left) finitely semisimple if each left finite-dimensional H-module is semisimple.

Examples of finitely semisimple Hopf algebras are:

1. $H = kG$, where G is a finite group with $(|G|, char\, k) = 1$.
2. $H = k[X]/(f)$, where k is of characteristic $p>0$, f is of the form $t_n X^{p^n} + t_{n-1} X^{p^{n-1}} + \cdots + t_0 X$ with $t_0 \neq 0$, and $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x = X + (f)$.
3. $H = U(L)$ – the enveloping algebra of a finite-dimensional, semisimple Lie algebra L over k (k is supposed to have characteristic 0).

Notice that if the Hopf algebra H is finitely semisimple and $H \otimes A \to A$ is a locally finite action of H on an algebra A, then every submodule of A, as an H-module, is semisimple.

The main purpose of this note is to prove the following.

Theorem 4. *Let $\gamma : H \otimes A \to A$ be an action of the Hopf algebra H on a local noetherian algebra (A,m) such that its maximal ideal m is H-invariant and $k \simeq A/m$. Then the action is linearizable if and only if the H surjection $p : m \to m/m^2$ admits an H retraction $t : m/m^2 \to m$ such that $pt = Id$. Such a t exists in each of the following cases:*

(1) There exists a semisimple H-submodule m' of A such that $Am' = m$.

(2) A is a complete local ring and the H-modules m/m^i, $i > 1$, are semisimple.

In particular, if H is finitely semisimple, then the action γ is linearizable if the action is locally finite or if the local ring A is complete.

Proof. First we show that the action γ is linearizable if there exists a homomorphism of H-modules $t : m/m^2 \to m$ such that $pt = Id$. For that purpose assume that such a t exists and choose $z_1,\ldots,z_n \in m$ which form a basis of m/m^2 over A/m. Then for any $h \in H$ and $i=1,\ldots,n$ $h.z_i = \sum_{j=1}^{j=n} \alpha_{ij} z_j$ for some $\alpha_{ij} \in k$, since $k \simeq A/m$. Set $x_i = t(z_i)$. Then we easily obtain that $h.x_i = h.t(z_i) = t(h.z_i) = t(\sum_{j=1}^{j=n} \alpha_{ij} z_j) = \sum_{j=1}^{j=n} \alpha_{ij} t(z_j) = \sum_{j=1}^{j=n} \alpha_{ij} x_j$, $i=1,\ldots,n$. By Nakayama Lemma the $x_1,\ldots,x_n$ form a minimal system of generators of the ideal m, because their images form a basis of m/m^2. This means that the action γ is linearizable. Conversely, if the action γ is linearizable, then the above consideration indicates how to define a homomorphism of H-modules $t : m/m^2 \to m$ such that $pt = Id$.

Now, to prove statement (1), suppose that m' is a semisimple H-submodule of A with $Am' = m$. Then the homomorphism of H-modules $p' : m' \to m/m^2$, $p'(x) = x + m^2$, is surjective (as $Am' = m$ and $k \simeq A/m$), and the short exact sequence of H-modules

$$O \to m' \cap m^2 \to m' \xrightarrow{p'} m/m^2 \to O$$

splits, since any submodule of a semisimple module is its direct summand. Hence, it follows that there is a homomorphism of H-modules $t' : m/m^2 \to m'$ with $p't' = Id$. Let $t : m/m^2 \to m$ be the composition of t' and the inclusion $m' \subset m$. Then clearly $pt = Id$, and thus, by the first part of the proof, statement (1) is proved. Now suppose that the local ring A is complete and that the H-modules m/m^j, $j > 0$, are semisimple. Similarly as above we need only to construct a homomorphism of H-modules $t : m/m^2 \to m$ with $pt = Id$. Let $p_j : m/m^{j+1} \to m/m^j$, $j \geq 2$, be the homomorphisms of H-modules defined by $p_j(x + m^{j+1}) = x + m^j$. Since A is complete, then, by [2, Proposition 10.13], the natural surjections $m \to m/m^{j+1}$, $j \geq 1$, induce an isomorphism of H-modules $m \simeq Lim_{\leftarrow}\{m/m^j m, p_{j+1}\}_{j \geq 1}$. Therefore, to construct a required homomorphism $t : m/m^2 \to m$, it is sufficient to find homomorphisms of H-modules $t_j : m/m^2 \to m/m^j m = m/m^{j+1}$, $j \geq 1$, such that $p_{j+1} t_{j+1} = t_j$ for all $j \geq 1$ and $t_1 = Id$. We proceed by induction on j. Set $t_1 = Id$ and assume that $t_1, \ldots, t_j$ have been defined for some $j \geq 1$. By semisimplicity of m/m^{j+2}, there is a homomorphism of H-modules $l : m/m^{j+1} \to m/m^{j+2}$ such that $p_{j+1} l = Id$. Hence if we put $t_{j+1} = l t_j$, then t_{j+1} is a homomorphism of H-modules with $p_{j+1} t_{j+1} = p_{j+1} l t_j = t_j$. This completes the proof of statement (2), and consequently Theorem 4 follows. □

Now we give a few consequences of Theorem 4.

In view of Examples 1–3, a direct consequence of Theorem 4 is the following.

Corollary 5. *Let (A, m) be a local algebra with $k \simeq A/m$.*

(1) For any finite group G of automorphisms of A with $(|G|, char\, k) = 1$ there exists a minimal system of generators $x_1, \ldots, x_n$ of the maximal ideal m such that $g(x_i) \in kx_1 + \cdots + kx_n$ for all $g \in G$ and $i = 1, \ldots, n$.

(2) If $char\, k = p > 0$ and $d : A \to A$ is a derivation of A such that $d(m) \subset m$ and d satisfies an equation $f(d) = 0$ with $f(X) = t_s X^{p^s} + t_{s-1} X^{p^{s-1}} + \cdots + t_0 X$, $t_i \in k$, $t_0 \neq 0$, then there exists a minimal system of generators $x_1, \ldots, x_n$ of m such that $d(x_i) \in kx_1 + \cdots + kx_n$ for $i = 1, \ldots, n$. Moreover, if the field k is algebraically closed, a minimal system of generators $x_1, \ldots, x_n$ can be chosen in such a way that $d(x_i) = \lambda_i x_i$, $i = 1, \ldots, n$, where all λ's are roots of the equation $f(X) = 0$ in k.

(3) Suppose that L is a finite-dimensional, semisimple Lie algebra over k, $char\, k = 0$, and $\lambda : L \to \mathrm{Der} A$ is a morphism of Lie algebras such that $\lambda(a)(m) \subset m$ for $a \in L$. Then there exists a minimal system of generators $x_1, \ldots, x_n$ of m such that $\lambda(a)(x_i) \in kx_1 + \cdots + kx_n$ for all $a \in L$ and $i = 1, \ldots, n$.

Remark. For $A = k[[Y_1, \dots, Y_n]]$ the second part of statement (2) follows from [1, Lemma (6.4)]. For $A = \mathbb{C}[[Y_1, \dots, Y_n]]$ and $L = sl(2, \mathbb{C})$ statement (3) was proved in [11, Proposition 2.1].

Corollary 6 (Well known). *Suppose that the field k is algebraically closed and V is an affine variety with a regular action of a linear algebraic group G (V and G defined over k). Moreover, suppose that $x \in V$ is such that the isotropy subgroup $G_x = \{g \in G,\ g.x = x\}$ is linearly reductive. Then the induced action of G_x on the local algebra $O_{V,x}$ is linearizable.*

Proof. Let $k[V]$ denote the algebra of the regular functions. Then the action of G on V induces a (locally finite) action of the Hopf algebra $H = kG_x$ on $k[V]$ and $O_{V,x} = k[V]_{m_x}$ (given by $(g.f)(v) = f(g^{-1}v)$ for $g \in G_x$, $f \in k[V]$ or $f \in O_{V,x}$, and $v \in V$). We obtain that m_x – the maximal (H-invariant) ideal in $k[V]$ corresponding to x – is a semisimple H-submodule of $O_{V,x}$, because G_x is linearly reductive. Furthermore, m_x generates the maximal ideal of $O_{V,x}$. The conclusion now follows, by part (1) of Theorem 4. □

Remark 7. Without any assumption on G_x there always exists a system of generators $f_1, \dots, f_r$ of $m_x O_{V,x}$ not necessarily minimal such that $g.f_i \in kf_1 + \cdots + kf_r$ for $i = 1, \dots, r$ and all $g \in G$. This is so since V can be G-equivariantly embedded as a closed subvariety of an affine n-space with a linear action.

In the characteristic zero case G_x is linearly reductive if G is linearly reductive and the orbit Gx is closed [6].

In order to formulate the next results, let us recall that given an action of the Hopf algebra H on an algebra A, $A^H = \{a \in A,\ \forall_{h \in H}\ h.a = \varepsilon(h)a\}$ is a subalgebra in A called the *algebra of invariants*. If V is a vector space over k, then an action of H on the symmetric (graded) algebra $S(V) = \bigoplus_{i \geq 0} S^i(V)$ is said to be *linear* if $h.V \subset V$ for each $h \in H$ or, equivalently, if $h.S^i(V) \subset S^i(V)$ for $h \in H$, $i \geq 0$. In particular, an action of H on the algebra of polynomials $k[X_1, \dots, X_n] = S(kX_1 + \cdots + kX_n)$ is linear if $h.X_i \in kX_1 + \cdots + kX_n$ for all $h \in H$ and $i = 1, \dots, n$. Obviously, any linear action of H on $S(V)$ is locally finite, whenever V is finite-dimensional, and $S(V)^H$ is a graded subalgebra of $S(V)$. Exactly in the same manner as for the rational actions of linear algebraic groups on algebras (see, e.g., [4, Ch. V]) one proves the following.

Proposition 8. *Assume that H is finitely semisimple and $H \otimes A \to A$ is a locally finite action of H on an algebra A.*

(1) *If A is noetherian, then the algebra of invariants A^H is also noetherian. Moreover, if $A = S(V)$, where V is of finite dimension, and the action is linear, then $S(V)^H$ is a finitely generated k-algebra.*

(2) *If H is cocommutative (i.e., $t\Delta = \Delta$, where $t: H \otimes H \to H \otimes H$ is a linear map given by $t(x \otimes y) = y \otimes x$) and A is finitely generated, then A^H is also finitely generated.*

Now we can prove.

Theorem 9. *Suppose that the Hopf algebra H is finitely semisimple, $A=k[[Y_1,\dots,Y_n]]$, and that H acts on A in such a way that the maximal ideal m of A is H-invariant.*

(1) There exist $X_1,\dots,X_n$ in m such that $A=k[[X_1,\dots,X_n]]$ and $h.X_j\in kX_1+\cdots+kX_n$ for $h\in H$, $j=1,\dots,n$.

(2) The natural H-module structure on m/m^2 extends (uniquely) to a linear action of H on the symmetric algebra $S(m/m^2)$ such that A^H is isomorphic to the completion of $S(m/m^2)^H$ in the topology defined by the powers of its irrelevant maximal ideal.

(3) A^H is a complete local noetherian ring.

Proof. Part (1) of the theorem is a consequence of part (2) of Theorem 4. As for part (2), notice that if $X_1,\dots,X_n$ are as in part (1), then the restriction of the action of H on A gives us a linear action of H on $k[X]=k[X_1,\dots,X_n]$. Hence, by means of the isomorphism of k-algebras $F: k[X]\to S(m/m^2)$ determined by $F(X_i)=X_i+m^2$, $i=1,\dots,n$, we may define a unique linear action of H on $S(m/m^2)$ such that $F(h.f)=h.F(f)$ for $h\in H$ and $f\in k[X]$. It follows that F induces an isomorphism of the completions of the graded algebras $k[X]^H$ and $S(m/m^2)^H$ in the topologies defined by the powers of the corresponding irrelevant maximal ideals. Therefore, to prove (2), it suffices to show that the algebra A^H is isomorphic to the completion of $k[X]^H$ in the $M=(X_1,\dots,X_n)\cap k[X]^H$-adic topology. It is clear that A^H is isomorphic to the completion of $k[X]^H$ in the topology defined by the degree. Moreover, from Proposition 8(1) we know that $k[X]^H$ is a finitely generated (graded) k-algebra. In view of [5, II.2.1, 6(vi)], this implies that the topology in $k[X]^H$ defined by the degree is equivalent to the M-adic topology. So, part (2) is proved. Part (3) of the theorem follows from the proof of part (2), since it has been shown above that A^H is the completion of the noetherian algebra $k[X]^H$ in the M-adic topology. □

An immediate consequence of part (2) of the above theorem applied to H from Example 1 is the following particular case of the main result of [9] (see also, [3, par. 5, Example 7]).

Corollary 10. *If G is a finite group of automorphisms of the algebra $k[[X]]=k[[X_1,\dots,X_n]]$ such that $(|G|, \operatorname{char} k)=1$ and the image of G under the induced homomorphism of groups $G\to GL(m/m^2)$, $m=(X_1,\dots,X_n)$, is a (finite) reflection group, then the algebra of invariants $k[[X]]^G$ is isomorphic to $k[[X]]$.*

Now let us assume that the field k is algebraically closed and V is an algebraic variety over k. Moreover, let G be a finite group acting (regularly) on V in such a way that each point of V is contained in an affine G-invariant subset of V (this assumption is verified in case V is quasiprojective). Then the space of orbits V/G has a natural structure of algebraic variety and the natural map $\pi: V\to V/G$ is a finite morphism of varieties, see [8, Ch. II, par. 7, Theorem 1]. Let $x \in V$ and let m_x be the maximal ideal in $O_{V,x}$. Then the action of G on V induces an action of the isotropy group G_x on the completion $\widehat{O}_{V,x}$ of the ring $O_{V,x}$ and a linear action of G_x on the vector space m_x/m_x^2. The latter action induces in turn an action of G on the algebra $S(m_x/m_x^2)$. In this situation the following holds.

Theorem 11. (1) *$\widehat{O}_{V/G,\pi(x)}$ is isomorphic to the algebra $(\widehat{O}_{V,x})^{G_x}$.*

(2) *If x is a regular point of V and $(|G_x|, \mathrm{char}\, k) = 1$, then $\widehat{O}_{V/G,\pi(x)}$ is isomorphic to the completion of $S(m_x/m_x^2)^{G_x}$ in the topology defined by the powers of its irrelevant (maximal) ideal.*

Proof. Part (1) is known in the case where $G_x = (e)$ [8, Ch. II, par. 7, Theorem 1]. The proof easily carries over to the general case. As for part (2), in view of regularity of x, $\widehat{O}_{V,x} \simeq k[[X_1,\dots,X_n]]$ (for some n), whence, using (1), $\widehat{O}_{V/G,\pi(x)} \simeq (\widehat{O}_{V,x})^{G_x} \simeq [[X_1,\dots,X_n]]^{G_x}$. The conclusion now follows from part (2) of Theorem 9 applied to $H = kG_x$. □

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